STABILITY OF CONVECTION FLOWS

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This paper concludes the analysis of the onset of instability in a fluid heated from below in a gravitational field, in the case of a prime eigen number.

It was shown in [1] that the problem of convection in a fluid layer has secondary stationary solutions, i. e. that bifurcation takes place. Paper [2] established that new stationary solutions occur, when an increasing temperature gradient reaches a critical value, and that in the case of the eigenvalue of a linearized problem being a prime number there are exactly two solutions.

It is shown here with the aid of the perturbation theory that the secondary motions are stable, whereas the equilibrium solution loses stability, when the critical value of temperature is reached (Sections 1 to 5). The index of nontrivial solutions (defined as fixed points of corresponding operator equations) is computed, and found to be equal to $+1$ (Section 6). Proof is also given (Section 7) that in the critical case the equilibrium solution is asymptotically stable (in a linear formulation there is stability, but it is not asymptotic.

Final conclusions are set out in Section 8, and a phase representation (Fig. 1) is given for small super-critical values of the temperature gradient of the system under consideration.

1. Formulation of the problem. Let a fluid fill a bounded region Ω. We shall assume that its boundary S is a solid wall (with no-slip condition fulfilled), the temperature of which is known, and is a linear function of height. Then, the convection equations

$$-\frac{\partial \mathbf{v}'}{\partial t}+\nu\triangle\mathbf{v}'=(\mathbf{v}'\cdot\nabla)\mathbf{v}'+\nabla p'+\beta T'\mathbf{g},\qquad \operatorname{div}\mathbf{v}'=0$$
$$-\frac{\partial T'}{\partial t}+\chi\triangle T'=\mathbf{v}'\cdot\nabla T',\qquad \mathbf{v}'|_S=0,\qquad T'|_S=cz+\text{const} \tag{1.1}$$

admit the solution

$$\mathbf{v}_0'=0,\qquad T_0'=cz+\text{const} \tag{1.2}$$

Let c_0 be the least eigenvalue of the corresponding linearized problem, and let (φ,τ) be its corresponding eigen solution

$$\nu\triangle\boldsymbol{\varphi}-\nabla q=\beta\tau\mathbf{g},\qquad \operatorname{div}\boldsymbol{\varphi}=0,\qquad \chi\triangle\tau=c_0\varphi_3,\qquad \tau|_S=0,$$
$$\boldsymbol{\varphi}|_S=0,\quad \|\boldsymbol{\varphi}\|_{H_1}=1 \tag{1.3}$$

For $c\leq c_0$, problem (1.1) has no stationary solutions other than (1.2) (see [2 to 4]), and all flows tend to the flow pattern (1.2), as $t\to\infty$ (*). As was shown in [2], when c

*) For Foot Note see next page.

reaches the value c_0, a pair of new stationary solutions of the form

$$\mathbf{v}_{0k} = \mp \alpha_0 \varepsilon \boldsymbol{\varphi} + (\alpha_0^2 \mathbf{w} + \beta_k \boldsymbol{\varphi}) \varepsilon^2 + O(\varepsilon^3) \qquad (\varepsilon = \sqrt{c - c_0})$$
$$T_{0k} = \mp \alpha_0 \varepsilon \tau + (\alpha_0^2 \theta + \beta_k \tau) \varepsilon^2 + O(\varepsilon^3) \qquad (k = 1, 2) \tag{1.4}$$

occur. Here, ε is a small parameter, and $\mathbf{w}$, θ is the solution of problem

$$\nu \Delta \mathbf{w} - \nabla p = (\boldsymbol{\varphi} \cdot \nabla) \boldsymbol{\varphi} + \beta \theta \mathbf{g}, \qquad \chi \Delta \theta = c_0 w_3 + \boldsymbol{\varphi} \cdot \nabla \tau$$
$$\operatorname{div} \mathbf{w} = 0, \qquad \mathbf{w}|_S = 0, \qquad \theta|_S = 0, \qquad (\mathbf{w} \cdot \boldsymbol{\varphi})_{H_1} = 0 \tag{1.5}$$

Constant α_0 is defined by the equality

$$\alpha_0 = \frac{1}{\sqrt{c_0 \gamma}}, \qquad \gamma = \|\mathbf{w}\|_{H_1}^2 + \frac{\beta g \chi}{\nu c_0} \|\theta\|_{H_2}^2 + \frac{2\beta g}{\nu} \int_\Omega \theta w_3 \, dx > 0 \tag{1.6}$$

The single-valued solvability of problem (1.5) and the positive nature of constant γ were proved in [2] (see [2] for Lemmas 2.1, 2.2 and 2.3). Constants β_1, β_2 may be considered as known, however, their explicit expressions have not been worked out here, as these do not matter in further considerations. The stability of flows (1.2) and (1.4) is studied further.

2. The perturbation theory. Let problem (1.1) have for $c = c_0 + \varepsilon^2$ and small ε, the stationary solution $(\mathbf{v}_0, T_0)$

$$\mathbf{v}_0 = \sum_{k=1}^{\infty} \varepsilon^k \mathbf{v}_k, \qquad T_0 = (c_0 + \varepsilon^2) z + T_{00}, \qquad T_{00} = \sum_{k=1}^{\infty} \varepsilon^k T_k \tag{2.1}$$

To solve the problem of stability of solution (2.1) we shall construct variation equations, and isolate time. As the result, we arrive at the spectral problem

$$-\sigma \mathbf{u} + \nu \Delta \mathbf{u} = (\mathbf{v}_0 \cdot \nabla) \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{v}_0 + \nabla p + \beta T \mathbf{g}, \qquad \operatorname{div} \mathbf{u} = 0$$
$$-\sigma T + \chi \Delta T = (c_0 + \varepsilon^2) u_3 + \mathbf{v}_0 \cdot \nabla T + \mathbf{u} \cdot \nabla T_{00}, \qquad \mathbf{u}|_S = 0, \quad T|_S = 0 \tag{2.2}$$

Problem (2.2) has its eigenvalue $\sigma_0 = 0$ when $\varepsilon = 0$, with all remaining eigenvalues contained within the left-hand half-plane. For small values of ε the latter, in accordance with the perturbation theory [10] are subject to little change, and remain in the left-hand half-plane.

Strictly speaking the perturbation theory is applicable to a limited part of the spectrum only, but, as in [11], the eigenvalues of problem (2.2) with a positive real part, are limited (uniformly with respect to ε when $|\varepsilon| \leq \varepsilon_0$), and their number is finite.

Thus, solution (2.1) will be stable, or unstable, depending on whether the eigenvalue $\sigma_0 = 0$ moves to the left, or right as the result of perturbation.

Let us look for the corresponding eigen solution of problem (2.2) in the form of a power series

$$\sigma = \varepsilon \sigma_1 + \varepsilon^2 \sigma_2 + \cdots, \qquad \mathbf{u} = \boldsymbol{\varphi} + \varepsilon \mathbf{u}_1 + \varepsilon^2 \mathbf{u}_2 + \cdots$$
$$T = \tau + \varepsilon \tau_1 + \varepsilon^2 \tau_2 + \cdots \tag{2.3}$$

*) A general theorem as to the existence and uniqueness of system (1.1) with initial data is not known. The theorem of existence of a weak generalized solution, and the theorem of the uniqueness of a smooth solution can be, however, proved. This is easily done by the methods developed in [5 and 6] (see also [7]): a two-dimensional problem, as well as the problem in which convection is disregarded, are solved as a whole [7 to 9]. Here, all statements about "all solutions" refer to generalized solutions.

It is natural to take the normalization condition in the form

$$(\mathbf{u}, \boldsymbol{\varphi})_{H_1} = \int\limits_{\Omega} \frac{\partial \mathbf{u}}{\partial x_k} \frac{\partial \boldsymbol{\varphi}}{\partial x_k} dx = 1 \qquad (2.4)$$

Substituting (2. 3) into (2. 2), we find the confirmation that φ, τ is a solution of system (1. 3), while $(\mathbf{u}_1, \tau_1, \sigma_1)$, $(\mathbf{u}_2, \tau_2, \sigma_2)$ is found by solving problems

$$\begin{gathered} \nu \Delta \mathbf{u}_1 = \nabla p_1 + \beta \tau_1 \mathbf{g} + R^\circ (v_1, \boldsymbol{\varphi}) + \sigma_1 \boldsymbol{\varphi}, \qquad \operatorname{div} \mathbf{u}_1 = 0 \\ \chi \Delta \tau_1 = c_0 u_{13} + \boldsymbol{\varphi} \cdot \nabla T_1 + \mathbf{v}_1 \cdot \nabla \tau + \sigma_1 \tau, \qquad \mathbf{u}_1 |_S = 0, \qquad \tau_1 |_S = 0 \quad (2.5) \\ (\mathbf{u}_1 \cdot \varphi)_{H_1} = 0 \end{gathered}$$

$$\begin{gathered} \nu \Delta \mathbf{u}_2 = \nabla p_2 + \beta \tau_2 \mathbf{g} + R^\circ (\mathbf{u}_1, \mathbf{v}_1) + R^\circ (\mathbf{v}_2, \boldsymbol{\varphi}) + \sigma_2 \boldsymbol{\varphi} + \sigma_1 \mathbf{u}_1 \\ \chi \Delta \tau_2 = c_0 u_{23} + \varphi_3 + \mathbf{v}_1 \cdot \nabla \tau_1 + \mathbf{v}_2 \cdot \nabla \tau + \mathbf{u}_1 \cdot \nabla T_1 + \boldsymbol{\varphi} \cdot \nabla T_2 + \sigma_2 \tau + \sigma_1 \tau_1 \quad (2.6) \\ \operatorname{div} \mathbf{u}_2 = 0, \qquad \mathbf{u}_2 |_S = 0, \qquad \tau_2 |_S = 0, \qquad (\mathbf{u}_2, \boldsymbol{\varphi})_{H_1} = 0 \end{gathered}$$

The following notation has been used here

$$R^\circ (\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \nabla) \mathbf{v} + (\mathbf{v}, \nabla) \mathbf{u}, \quad \mathbf{u}_k = (\mathbf{u}_{k1}, \mathbf{u}_{k2}, \mathbf{u}_{k3}), \quad R (\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \nabla) \mathbf{v}$$

3. Stability of secondary flows. When dealing with solutions (1, 4), we must stipulate that in (2, 1)

$$v_1 = \mp \alpha_0 \varphi, \quad v_2 = \alpha_0^2 w + \beta_k \varphi, \quad T_1 = \pm \alpha_0 \tau, \quad T_2 = \alpha_0^2 \theta + \beta_k \tau \qquad (3.1)$$

We shall prove that with this, the solution of problem (2. 5) has the form

$$\sigma_1 = 0, \qquad u_1 = \mp 2\alpha_0 w, \qquad \tau_1 = \mp 2\alpha_0 \theta \qquad (3.2)$$

In fact, taking the scalar product of the first Eqs. (2. 5) by $c_0 \varphi$, and for the second by $\beta g \tau$, then integrating over Ω and adding, we obtain

$$\sigma_1 \left[c_0 \int\limits_{\Omega} \varphi^2 \, dx + \beta g \int\limits_{\Omega} \tau^2 \, dx \right] = 0$$

Therefore, $\sigma_1 = 0$, and (3. 2) follows directly from (1. 5) and (2. 5).

Furthermore, by dealing in a likewise manner with system (2. 6), we obtain

$$-\sigma_2 I_0 = c_0 I_1 + \beta g (I_2 + I_3) \equiv I \qquad (3.3)$$

$$I_0 = c_0 \int\limits_{\Omega} \varphi^2 \, dx + \beta g \int\limits_{\Omega} \tau^2 \, dx, \quad I_1 = \int\limits_{\Omega} [(v_1, \nabla) u_1 + (\boldsymbol{\varphi}, \nabla) \mathbf{v}_2] \cdot \boldsymbol{\varphi} \, dx$$

$$I_2 = \int\limits_{\Omega} \tau [\mathbf{v}_1 \cdot \nabla \tau_1 + \boldsymbol{\varphi} \cdot \nabla T_2] \, dx, \qquad I_3 = \int\limits_{\Omega} \varphi_3 \tau \, dx$$

With the aid of (3. 1) and (3. 2) we derive.

$$I_1 = -3\alpha_0^2 \int\limits_{\Omega} (\boldsymbol{\varphi}, \nabla) \boldsymbol{\varphi} \cdot \mathbf{w} \, dx, \qquad I_2 = -3\alpha_0^2 \int\limits_{\Omega} \theta \boldsymbol{\varphi} \cdot \nabla \tau \, dx \qquad (3.4)$$

Taking the scalar product of the first of Eqs. (1. 3) by $\boldsymbol{\varphi}$, and integrating over Ω, we find

$$I_3 = \int\limits_{\Omega} \varphi_3 \tau \, dx = -\frac{\nu}{\beta g} \qquad (3.5)$$

Now, by substituting in (3. 4) for $(\boldsymbol{\varphi}, \nabla) \boldsymbol{\varphi}$ and $\boldsymbol{\varphi} \cdot \nabla \mathbf{T}$ their expressions obtained from

(1, 5), and taking into account (3, 5) and (1, 6), we arrive at

$$I = 3\alpha_0^2 c_0^2 \qquad J(\mathbf{w},\ \theta) + \beta g I_3 = 2\nu \tag{3.6}$$

Hence, the eigenvalue $\sigma_0 = 0$, after perturbation, is transformed into

$$\sigma = -2\nu\varepsilon^2 / I_0 + O(\varepsilon^3) < 0 \quad (\epsilon \text{ is small}) \tag{3.7}$$

This proves that secondary motions (1, 4) are asymptotically stable in a linear approximation. But in essence, the results obtained in [11] (with obvious alterations) are applicable to problem (1, 1). Therefore, a nonlinear stability is also obtained.

In the case of convection in a layer, the secondary flow stability with respect to perturbations of like periodicity, follows from the foregoing. It may be thought that only an analysis of the effect of nonperiodic perturbations would show which of these flows can be obtained experimentally. This is the obvious path leading to the resolution of the question of the exceptional role of the hexagonally-symmetric flows.

4. Instability of equilibrium. We shall now apply the perturbation theory to the problem of stability of solution (1, 2). In this case the eigenvalue $\sigma_0 = 0$ is transformed by perturbations and becomes

$$\sigma = \nu\varepsilon^2 / I_0 + O(\varepsilon^3) > 0 \tag{4.1}$$

For the derivation of Formula (4, 1) it is obviously sufficient to assume that in (3, 6) $\alpha_0 = 0$.

Thus, when parameter c passes through its critical value c_0, the equilibrium solution (1, 2) loses stability. Here this deduction is also justified for the case of the nonlinear system (1, 1) with the aid results of [11]. In [4] the instability was proved by another method of linear approximation.

5. Proof of the perturbation theory. Problem (2, 2) will be reduced by transforming the Navier-Stokes linearized operator and the Laplace operator to the system of equations

$$\mathbf{u} = L\ \beta T \mathbf{g}) + LR^\circ(\mathbf{v}_0, \mathbf{u}) + \sigma L\mathbf{u} \tag{5.1}$$

$$T = c_0 B_0 u_3 + \varepsilon^2 B_0 u_3^* + B_0(\mathbf{v}_0 \cdot \nabla T + \mathbf{u} \cdot \nabla T_{00}) + \sigma B_0 T$$

Operators L and B_0 are defined in more detail in [1]. Operator L acts fully continuously from $\mathbf{L}_p\ (p > {}^6/_5)$ into H_1, and operator B_0 from $L_p\ (p > {}^6/_5)$ into H_2.

We eliminate T from this system. By virtue of (2, 1) the operators at the right-hand side of (5, 1) depend analytically on ϵ (for example, with respect to the norms of H_1, H_2). Further to this, for small ϵ, σ each of these is a contraction operator (for $\epsilon = 0$, $\sigma = 0$ both are reduced to constants) in H_1 and H_2, respectively. Therefore, the solution of the second of Eqs. (5, 1) with small ϵ, σ and a fixed $u \in H_1$ may be sought in the form of a power series

$$T = \sum_{k,\ l=0}^{\infty} \varepsilon^k \sigma^l \theta_{kl} \tag{5.2}$$

In H_2, series (5, 2) is convergent. Substituting it into (5, 1), we obtain

$$\theta_{00} = c_0 B_0 u_3, \qquad \theta_{10} = B_0(\mathbf{v}_1 \cdot \nabla\theta_{00} + \mathbf{u} \cdot \nabla T_1)$$

$$\theta_{01} = B_0\theta_{00} = c_0 B_0^2 u_3,$$

$$\theta_{20} = B_0(v_1 \cdot \nabla\theta_{10} + v_2 \cdot \nabla\theta_{00} + u \cdot \nabla T_2) + B_0 u_3 \tag{5.3}$$

After the substitution of series (5, 2) into the first of Eqs. (5, 1), the latter becomes

$$\mathbf{u} = c_0 A\mathbf{u} + N\mathbf{u}, \quad A\mathbf{u} = L(\beta \mathbf{g} B_0 u_3), \quad N\mathbf{u} = \sum_{k,l=0}^{\infty} \varepsilon^k \sigma^l N_{kl}\mathbf{u}, \quad N_{00} = 0 \tag{5.4}$$

With this, operator A is fully continuous and rigorously positive in H (see [1 and 3]), operator N is fully continuous in H_1 and analytically dependent on ϵ, σ (when these are small). Explicit expressions of operator-coefficients N_{kl} can be derived without difficulty. We have, for example

$$N_{10}\mathbf{u} = LR^{\circ}(\mathbf{v}_1, \ \mathbf{u}) + L(\beta \mathbf{g} \theta_{10}), \qquad N_{01}\mathbf{u} = L\mathbf{u} + c_0 L(\beta g B_0^2 u_3) \tag{5.5}$$

$$N_{20}\mathbf{u} = L(\beta \theta_{20} \mathbf{g}) + LR^0(\mathbf{v}_2, \ \mathbf{u})$$

We shall consider now Eqs. (5.4) for a specified small ϵ as a problem in eigenvalues with respect to the nonlinear parameter σ. In the following Lemma the specific nature of operators A, N is immaterial.

Lemma 5.1. Let A be a linear, fully self-adjoint operator in the Hilbert space H_1, c_0 its prime characteristic number and φ its corresponding eigenvector. Let operator N, continuous in H_1, depend analytically on the small parameters ϵ, σ. Let condition

$$(N_{01}\varphi, \ \varphi)_{H_1} \neq 0 \tag{5.6}$$

be fulfilled.

Then, for small ϵ problem (5.4) has a unique small eigenvalue σ which, like its corresponding eigenvector $\mathbf{u}$ (subject to condition that $(\mathbf{u}, \varphi)_{H_1} = 1$), is analytically dependent on ϵ.

Proof. Problem (5.4) may be rewritten in the following equivalent form

$$\mathbf{u} - c_0 A\mathbf{u} = N\mathbf{u} - (N\mathbf{u}, \varphi)_{H_1}\varphi \equiv N_0\mathbf{u}, \qquad (\mathbf{u}, \varphi)_{H_1} = 1 \tag{5.7}$$

$$(N\mathbf{u}, \varphi)_{H_1} = 0 \tag{5.8}$$

Then, in accordance with the Fredholm solvability condition, operator N_0 transfers any vector $\mathbf{u} \in H_1$ into a subspace where the inverse operator $(I - c_0 A)^{-1} = R_0$, identically fixed by the requirement that $(R_0 \mathbf{u}, \varphi)_{H_1} = 0$ is known. Hence, conditions (5.7) are equivalent to Eq.

$$\mathbf{u} = \varphi + (I - c_0 A)^{-1} N_0 \mathbf{u} \tag{5.9}$$

Since N_0 is analytically dependent on ϵ, σ, and $N_0 = 0$ when $\epsilon = \sigma = 0$, the right-hand side of (5.9) defines the contraction operator for small ϵ, σ. Solution $\mathbf{u}$ of Eq. (5.9) is, therefore, analytical with respect to ϵ, σ, and is of the form

$$\mathbf{u} = \sum_{r,s=0}^{\infty} \varepsilon^r \sigma^s \mathbf{u}_{rs}, \qquad \mathbf{u}_{00} = \varphi \tag{5.10}$$

Substituting (5.10) into (5.8) we obtain Eq.

$$F(\sigma, \varepsilon) \equiv \sum_{k,l,r,s=0}^{\infty} \varepsilon^{k+r} \sigma^{l+s} (N_{kl}\mathbf{u}_{rs}, \varphi)_{H_1} = 0 \tag{5.11}$$

which for a specified ϵ is satisfied by σ.

Here, $F(\sigma, \epsilon)$ is an analytical function, and $F(0, 0) = 0$.

Solution σ of Eq. (5.11) is unique and analytical with respect to ϵ, since conditions of the theorem of the implicit function

$$\left.\frac{\partial F}{\partial \sigma}\right|_{\epsilon, \sigma = 0} = (N_{01}\varphi, \varphi)_{H_1} \neq 0 \tag{5.12}$$

is satisfied.

The analytical character of vector $\mathbf{u}$ with respect to ϵ now follows from (5.10). Lemma is proved.

We may note that Lemma 5.1 is also valid for not-self-adjoint operators (as well as for operators in a Banach space), if we substitute for the second factor in (5.12) the eigenvector of the adjoint equation. Under the conditions of this Lemma the number σ is real.

We shall now prove that condition (5.12) is fulfilled in the case of our problem. We note that operator L satisfies by definition the following identity:

$$\nu (L\mathbf{f}, \mathbf{\Phi})_{H_1} \equiv - \int_\Omega \mathbf{f}\mathbf{\Phi} dx \qquad \left(\mathbf{f} \in L_p\left(p \geqslant \frac{6}{5}\right), \mathbf{\Phi} \in H_1\right) \tag{5.13}$$

Consequently, (5.5) and (5.13), when account is taken of the self-adjointness of operator B_0, yield

$$\nu (N_{01}\boldsymbol{\varphi}, \boldsymbol{\varphi})_{H_1} = - \int_\Omega \boldsymbol{\varphi}^2 dx - \beta g c_0 \int_\Omega (B_0 \varphi_3)^2 dx = - I_0 / c_0 < 0 \tag{5.14}$$

In accordance with Lemma 5.1 the existence of expansions (2.3) follows from (5.14). The perturbation theory is thus proved.

6. Indices of solutions. Stationary solutions of problem (1.1) satisfy the operator equation in space H_1 with a fully continuous operator (see [1 to 3])

$$\mathbf{v} = K(\mathbf{v}, c) \tag{6.1}$$

We shall show that the indices of solutions (1.2) and (1.4) representing fixed points of operator K are respectively -1 and $+1$. The knowledge of these indices may be useful in, for example, evaluating the number of solutions.

For the computation of the index of a certain solution $\mathbf{v}_0$ of Eq. (6.1) the Frechet differential $A_{\mathbf{v}_0}$ of operator K at point $\mathbf{v}_0$ must be considered, and the sum of multiplicities Δ of its characteristic numbers, lying on segment (0, 1) must be calculated. If 1 is not a characteristic number, then the index of the fixed point $\mathbf{v}_0$ is $(-1)^\Delta$ (see [12]).

The Frechet differential of operator K which corresponds to solution (1.2) is cA (operator A is defined by (5.4)). When $c > c_0$, and $c - c_0$ is small, 1 is not its characteristic number. In this case the unique characteristic number along segment (0, 1) is c_0/c. It is a prime number: $\Delta = 1$. Hence, the index of solution (1.2) is -1.

We shall now prove that the index of each of the solutions (1.4) is $+1$. The Frechet differential $A_{\mathbf{v}_0}$ is of the form

$$A_{\mathbf{v}_0}\mathbf{u} = c_0 A\mathbf{u} + \sum_{k=1}^{\infty} \varepsilon^k N_{k0}\mathbf{u} \tag{6.2}$$

which may be derived by stipulating $\sigma = 0$ in (5.4). For small values of ϵ operator $A_{\mathbf{v}_0}$ is close to $c_0 A \mathbf{u}$. Consequently, in accordance with the perturbation theory [10] it can't have characteristic numbers along segment (0, 1) other than those obtained by perturbation of the characteristic number 1 of operator $c_0 A$.

We shall show, however, that the latter lies outside segment (0, 1). We shall denote it by λ, and the corresponding eigenvector by ψ, and shall seek expressions of these in the form

$$\lambda = 1 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \quad \boldsymbol{\psi} = \boldsymbol{\varphi} + \varepsilon\boldsymbol{\psi}_1 + \varepsilon^2\boldsymbol{\psi}_2 + \cdots, \quad (\boldsymbol{\psi}, \boldsymbol{\varphi})_{H_1} = 1 \tag{6.3}$$

This is permissible, since 1 is a prime characteristic number of operator $c_0 A$, and so is λ. We substitute (6.3) into

$$\boldsymbol{\psi} = \lambda A_{\mathbf{v}_0}\boldsymbol{\psi} = \lambda \left(c_0 A\boldsymbol{\psi} + \sum_{k=1}^{\infty} \varepsilon^k N_{k0}\boldsymbol{\psi}\right) \tag{6.4}$$

We then obtain for the definition of (λ_1, ψ_1) the following Eq.

$$\psi_1 = c_0 A\psi_1 + N_{10}\varphi + \lambda_1\varphi, \qquad (\psi_1, \varphi)_{H_1} = 0 \tag{6.5}$$

Taking into account (5, 5), (5, 13), (5, 3) and (3, 1) and the scalar product of (6, 5) by $\nu\varphi$, we derive $\lambda_1 \nu c_0 = -\nu(N_{10}\varphi, \varphi)_{H_1} = -\nu(LR^\circ(\mathbf{v}_1, \varphi) + L(\beta g\theta_{10}), \varphi)_{H_1} =$

$$= \int_\Omega [R^\circ(\mathbf{v}_1, \varphi) + \beta g\theta_{10}]\varphi dx \mp \alpha_0\beta g \int_\Omega \varphi_3 B_0(c_0\varphi\cdot\nabla B_0\varphi_3 + \varphi\cdot\nabla\tau)\, dx =$$

$$= \mp \alpha_0\beta g \int_\Omega B_0\varphi_3\varphi\cdot\nabla\tau\, dx = 0 \tag{6.6}$$

The following relationships have also to be taken into account here

$$\tau = c_0 B_0\varphi_3, \qquad \varphi = c_0 A\varphi, \qquad \|\varphi\|_{H_1} = 1 \tag{6.7}$$

Hence, $\lambda_1 = 0$. We note now that

$$N_{10}\varphi = \mp 2\alpha_0 L[(\varphi, \nabla)\varphi + \beta g B_0(\varphi\cdot\nabla\tau)] \tag{6.8}$$

Comparing (6, 5), (6, 8) and (1, 5), we obtain

$$\psi_1 = \mp 2\alpha_0 \mathrm{w} \tag{6.9}$$

Vector ψ_2 and number λ_2 are defined by Eq.

$$\psi_2 = c_0 A\psi_2 + N_{10}\psi_1 + N_{20}\varphi + \lambda_2\varphi, \qquad (\psi_2, \varphi)_{H_1} = 0 \tag{6.10}$$

Taking the scalar product of this Eq. by $\nu\varphi$ in H_1, and using consecutively relationships (5, 13), (5, 5), (3, 1), (6, 9), (6, 7), (5, 3) and (1, 6), we obtain after a straightforward, though somewhat cumbersome computation

$$\lambda_2 = 2/c_0 > 0 \tag{6.11}$$

Hence, operator $A_{\mathbf{v}_0}$ has no characteristic numbers along segment (0, 1), $\Delta = 0$, and the index of each of the solutions (1, 4) is equal to +1.

7. Asymptotic stability in the critical case. We shall prove that when $c = c_0$, the equilibrium solution (1, 2) is generally asymptotically stable. We note that in this case there exists stability of the linearized problem, but it is not an asymptotic stability.

Multiplying the first Eq. of (1, 1) by $c_0\mathbf{v} = c_0(\mathbf{v}' - \mathbf{v}_0')$, and the second by $\beta gT = \beta g(T' - T_0')$, integrating over Ω, and adding, we obtain the following relationships for perturbations $\mathbf{v}, T$:

$$\frac{dJ_0(\mathbf{v}, T)}{dt} = -J(\mathbf{v}, T), \qquad J_0(\mathbf{v}, T) = \frac{1}{2}\int_\Omega\left(\mathbf{v}^2 + \frac{\beta g}{c_0}T^2\right)dx \tag{7.1}$$

$$J(\mathbf{v}, T) = \nu\|\mathbf{v}\|^2_{H_1} + \frac{\beta g\chi}{c_0}\|T\|^2_{H_2} + 2\beta g\int_\Omega Tv_3\, dx$$

We stipulate that in (7, 1)

$$\mathbf{v} = \mathbf{u} + a\varphi, \qquad T = R + a\tau \tag{7.2}$$

We shall define here parameter $a = a(t)$ from condition

$$\int_\Omega (c_0\mathbf{u}\cdot\varphi + \beta g R\tau)\, dx = 0 \tag{7.3}$$

Substituting (7, 2) into (7, 1), and using (1, 3), we obtain

$$\frac{d}{dt}[J_0(\mathbf{u}, R) + a^2 J_0(\varphi, \tau)] = -J(\mathbf{u}, R) \tag{7.4}$$

Lemma 7.1. For any $\mathbf{u} \in H_1, R \in H_2$, which satisfy condition (7.3) the following inequality is valid:

$$J(\mathbf{u}, R) \geqslant mJ_0(\mathbf{u}, R) \tag{7.5}$$

where constant $m > 0$ is independent of $\mathbf{u}, R$.

Proof. The functional

$$J_3(\mathbf{u}, R) = \int_\Omega R u_3 \, dx \tag{7.6}$$

is weakly continuous in $H_3 = H_1 + H_2$, because of the full continuity of embedding of H_1, H_2 in L_2. Therefore relationship $J_3 / J - 2\beta g J_3$ reaches a positive maximum m_0 in the subspace of space H_3, defined by condition (7.3). Since $J(\mathbf{v}, T) \geqslant 0$, with equality obtaining only with $\mathbf{v} = \alpha\varphi$, $T = \alpha\tau$, $\alpha = \text{const}$ (see [2]), therefore

$$J_3 / J - 2\beta g J_3 \leqslant {}^1/_2\, \beta g \tag{7.7}$$

The equality in (7.7) is reached only when $\mathbf{v} = \alpha\varphi$, $T = \alpha\tau$, $\alpha = \text{const}$. Consequently $m_0 < {}^1/_2\, \beta g$.

We now obtain inequality

$$J(\mathbf{u}, R) \geqslant (1 - 2\beta g m_0)\left(\nu \|\mathbf{u}\|_{H_1}{}^2 + \frac{\beta g \chi}{c_0} \|R\|_{H_2}{}^2\right) \tag{7.8}$$

from which we deduce directly inequality (7.5) by means of the embedding theorem. Lemma is proved.

The following evaluation is deduced from equality (7.4) and Lemma 7.1

$$J_0(\mathbf{u}, R) \leqslant e^{-mt} J_0(\mathbf{u}_0, R_0), \quad \mathbf{u}_0 = \mathbf{u}|_{t=0}, \; R_0 = R|_{t=0} \tag{7.9}$$

Multiplying (7.4) by $\exp m_1 t$, integrating from 0 to ∞ with respect to t, and using (7.9), we find

$$\int_0^\infty J \cdot (\mathbf{u}, R)\, e^{m_1 t}\, dt \leqslant \frac{m}{m - m_1} J_0(\mathbf{u}_0, R_0) \qquad m_1 < m \tag{7.10}$$

We shall now evaluate function $a(t)$ from (7.2). We substitute the following expressions into (1.1):

$$\mathbf{v}' = \mathbf{v} = \mathbf{u} + a\varphi, \qquad T' = cz + T = cz + R + a\tau \tag{7.11}$$

Taking into account Eqs. (1.3) and (7.3), we multiply the obtained Eqs. respectively by φ and $\beta g \mathrm{T}/c_0$; integrating over Ω, then adding, we obtain

$$\frac{da}{dt} = Ma + N, \qquad \begin{pmatrix} M = -J_0((\varphi, \nabla)\mathbf{u}, \varphi \cdot \nabla R) / J_0(\varphi, \tau) \\ N = -J_0((\mathbf{u}, \nabla)\mathbf{u}, \mathbf{u} \cdot \nabla R) / J_0(\varphi, \tau) \end{pmatrix} \tag{7.12}$$

Integration by parts yields the following expressions for parameters M, N.

$$\begin{aligned} M &= \frac{1}{J_0(\varphi\tau)} \int_\Omega \left[(\varphi, \nabla)\, \varphi \cdot \mathbf{u} + \frac{\beta g}{c_0}\, \varphi \cdot \nabla \tau R\right] dx \\ N &= \frac{1}{J(\varphi, \tau)} \int_\Omega \left[(\mathbf{u}, \nabla)\, \varphi \cdot \mathbf{u} + \frac{\beta g}{c_0}\, \mathbf{u} \cdot \nabla \tau R\right] dx \end{aligned} \tag{7.13}$$

The following estimates are obtained directly from (7.13) and (7.9)

$$\begin{aligned} M^2 &\leqslant m_2 J_0(\mathbf{u}, R) \leqslant m_2 e^{-mt} J_0, (\mathbf{u}_0, R_0) \\ |N| &\leqslant m_3 J_0(\mathbf{u}, R) \leqslant m_3 e^{-mt} J_0(\mathbf{u}_0, R_0) \end{aligned} \tag{7.14}$$

Constants m_2, m_3 are independent of $\mathbf{u}, R$.

Expressing a in terms of M and N from (7.12), and taking into consideration (7.14), we obtain the confirmation that with $t \to \infty$ $a(t)$ tends to a certain limit a_∞. It can be easily shown that, if a certain solution $(\mathbf{v}', T')$ of problem (1.1) has a limit in the sense of L_2 when $t \to \infty$, then the latter is a stationary solution of this problem. From this it follows, by virtue of (7.9) that $(a_\infty \varphi,\ c_0 z + a_\infty \tau)$ is a stationary solution. But, as was shown in [2], there are no nontrivial stationary solutions when $c = c_0$. Therefore, $a_\infty = 0$.

It has been thus proved that with $t \to \infty$, all solutions of problem (1.1) tend to the equilibrium solution (1.2).

This result may be somewhat refined. Namely, by stipulating for the decrease of coefficient $a(t)$ an asymptotic behavior of a power kind, which proves to be

$$a(t) \sim a_0 (1 + 2\delta a_0^2 t)^{-1/2} \qquad (t \to \infty), \qquad a_0 = a(0), \qquad \delta = \gamma / J_0(\varphi, \tau) \qquad (7.15)$$

Constant γ was defined in (1.6) and is positive.

8. Conclusion. We shall formulate here the obtained results which together give a full qualitative description of the first loss of stability in a convection problem for the case in which the eigenvalue c_0 is a prime number. A number of examples in which the condition of primality occurs have been analyzed in [1 to 3] (a spatially periodic problem, convection in a horizontal layer, convection in a long vertical cylinder).

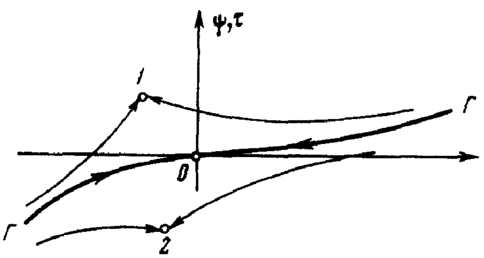

Fig. 1

1. The stationary solution (1.2) of problem (1.1) is unique when $c \leq c_0$, and all solutions of problem (1.1) tend to it when $t \to \infty$.

These facts were established in [2 to 4]; proof of the asymptotic stability in the critical case was given above.

2. For small positive $c - c_0$ there exist exactly two secondary stationary flows (1.4) which are asymptotically stable. The equilibrium solution (1.2) in this case loses its stability.

With the use of results of [11], we arrive at the pattern in a phase space (a point of which is the pair $(\mathbf{v}, T)$), shown on Fig. 1: multiplicity Γ of co-dimension 1 divides it into two "curved subspaces", each containing one of each points 1 and 2, attracting all the trajectories passing through these. The trajectories which originate on Γ tend to an equilibrium solution. A projection of this pattern onto a plane spanning the eigenvector (φ, T) and a certain other vector orthogonal to it, is shown in Fig. 1. Arrows indicate the direction of motion of points along their trajectories with increasing time t.

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